ASSIGNMENT SET – I

Mathematics: Semester-I

M.Sc (CBCS)

Department of Mathematics

Mugberia Gangadhar Mahavidyalaya

PAPER - MTM-102

Paper: Complex Analysis

Answer all the questions

1. Let $f(z) = z^4 + z^2 + 8z - 4$. Prove that exactly three roots of the polynomial f(z) lie in 1<|z|<3, and there is exactly one root of the polynomial f(z) in the disc $|z+2| < \frac{1}{2}$.
2. *f(z) = u(x, y) + i v(x, y)* in a domain D, where v is a harmonic conjugate of u and u is also a harmonic conjugate of v. Then show that f(z) is constant throughout in D.
3. State and proof the Cauchy's Residue theorem.
4. Using the method of residue formula evaluate $\int_{|z|=3} \frac{sin\pi z^2+cos\pi z^2}{(z-1)(z-2)}$.
5. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

6. Show that if *f(z)* is analytic within and on a simple closed contour C and z_0 is not on C, then
7. $\int_C \frac{f'(z)}{z-z_0} dz = \int_C \frac{f(z)}{(z-z_0)^2} dz$
8. Define Bilinear transformation. Characteristic the set of all Bilinear transformation which maps the upper half plane onto the interior unit disk.

9. Define conformal mapping. Is bilinear transformation a conformal mapping? justifies your answers.
10. Using the calculus of residues evaluate $\int_0^\infty \frac{dx}{(x^2+4)^3}$.
11. Let a function is defined by $f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, z \neq 0 \\ 0, z = 0 \end{cases}$. Show that the C-R equations are satisfied at (0,0) although f(z) is not differentiable at (0,0).
12. Characteristic all analytic functions f(z) whose real part is given by

    u(x,y) =log($\sqrt{x^2+y^2}$)
13. Find the upper bound for $\int_c \frac{1}{10+z^2}$dz, where c= z(t) = $2e^{it}$, -π≤ $t < \pi$.
14. Show that the curve C parameterized by

    z(t)=$\begin{cases} t(\cos(\frac{1}{t}) + i\sin(\frac{1}{t})) & \text{if } 0< t \leq 1, \\ 0 & if\ t = 0 \end{cases}$ is non rectifiable.
15. Give an example of a Harmonic function.
16. Using the calculus of residue evaluate $\int_0^\infty \frac{sinx}{x} dx$.
17. With the help of residue, find the inverse Laplace transformation f(t) of $F(s) = \frac{s}{(s^2+a^2)^2} (a > 0)$.
18. Find the singular points of the function $f(z) = \frac{Log(z+2+3i)}{z^4+1}$.
19. Find the Mobious transformation that maps $0, 1, \infty$ to the respective points $-i, \infty, 1$.
20. Write Taylor's and Laurent's series representation of a function f(z) by stating necessary condition/s for each of the series. Hence discuss under what condition/s the Laurent's series of the said function reduces to Taylor's series of the said function.
21. Show that $f(z) = z\bar{z}$ is nowhere analytic.
22. Is it possible to evaluate the integral $\int_C f(z)dz$, where $f(z) = \frac{(5z+2)}{(z(z-2))}$ and C: $|z| = 1$, using the single residue of $\frac{1}{z^2} f(\frac{1}{z})$ at $z = 0$? Justify.
23. State the Laurent's theorem.
24. Find the essential singularities of $f(z) = e^{\frac{1}{z}}$.
25. Let C be any simple closed contour, described the positive sense in the z-plane and write $g(w) = \int_C \frac{2z+z^3}{(z-w)^3} dz$. Then find $g(w)$ when w is inside C.

26. What is the condition for the existence of conformal mapping of the function
$f(z) = \frac{az+b}{cz+d}$?

27. Without evaluating , find an upper bound of the integral $\int_C \frac{e^{2z} - \frac{\sqrt{3}\,\bar{z}}{2}}{z^2+2} dz$

, where C

is the arc of the circle $|z| = \sqrt{3}$ from $z = -\sqrt{3}$ to $z = -i\sqrt{3}$ taking in anti-clockwise

direction.

28. If $(z) = u + iv$, is an analytic function and $u = e^{-x}(x \sin y - y \cos y)$, then find v.
29. State and prove the Rouche's theorem
30. Find the order of the pole at $z = \frac{\pi}{4}$ of the function $f(z) = \frac{1}{\cos z - \sin z}$.
31. If C is a regular arc joining the $A(z = a)$ to $B(z = b)$ then show that $\int_{AB} z dz = \frac{b^2 - a^2}{2}$
32. Using the method of residues, evaluate : $\int_0^\infty \frac{x \sin mx}{x^2+a^2} dx$
33. State and prove Liouville's theorem.
34. Find a conformal map of the unit disk $|z| < 1$ onto the right half-plane $Re(w) > 0$.
35. Find the total number of zeros of a function $F(z) = z^8 - 5z^5 - 2z + 1$ inside the circle $|z| < 1$.
36. State the Jordan's Lemma.
37. Evaluate : $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 10} dx$, using the method of residue only.
38. Sketch $S = \{z: \left|\frac{z+1}{z-1}\right| < 1\}$ and decide whether it is domain.
39. Find the value of x and y for which $\sin(x + iy) = \cosh(4)$.
40. Show that the function $f(z) = \frac{z}{e^z - 1}$ has a removable singularity at the origin.
41. Evaluate: $\int_{|z|=1} z\bar{z}\ dz$.

42. Find the coefficient of $\frac{1}{z^3}$ in the Laurent series expansion of $f(z)$ for $z > 2$ where $f(z) = \frac{1}{z^2-3z+2}$.

43. Find the Mobious transformation that maps $0, 1, \infty$ to the respective points $0, i, \infty$.

44. Is the function $u = 2xy + 3xy^2 - 2y^3$ is harmonic?

45. Suppose that $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then prove that the first order partial derivatives of u and v must exists at (x_0, y_0) and they must satisfy the Cauchy Riemann equations: $u_x = v_y$; $u_y = -v_x$ at (x_0, y_0). Also prove that $f'(z_0) = u_x + iv_x$ at (x_0, y_0).

46. Evaluate the integral $\int_C \frac{f(z)+f(-1/z)}{(z-i)^2}\, dz$ where C is the simple closed contour $|z-i| = \frac{1}{2}$, in counter clockwise sense and $f(z)$ is analytic in $|z-i| \leq 1$.

47. Define Meromorphic function. If $f(z) = u(x,y) + iv(x,y)$ and $g(z) = v(x,y) + ik(x,y)$ be non-zero analytic functions on $|z| < 1$. Then prove that $f(z)$ is a constant function.

48. Find the real part of the analytic function whose imaginary part is $x^2 - y^2 + \frac{x}{x^2+y^2}$. Also find the analytic function.

49. Using the method of residues: Evaluate $\int_0^{2\pi} \frac{d\phi}{a+\cos\phi}, a > 1$.

50. Define zero's of an analytic function. Find the zero's of the function $f(z) = coshz$.

51. State and prove the Cauchy's Residue Theorem.

52. State and prove the Jordon's Lemma.

53. State Cauchy's Integral Theorem. Using the method of residue calculus, Evaluate: $\int_0^\infty \frac{\sin x}{x^p}\, dx, 0 < p < 1$.

54. Let $w = f(z) = \frac{\mathrm{az+b}}{cz+d}$ is a bilinear transformation. Then find the inverse of the transformation. Is it again a bilinear? Also find the determinant of both the transformations.

55. Use the Schwartz-Chritoffel transformation to arrive at the transformation $w = z^m$ $(0 < z < 1)$, which maps the half plane $y \geq 0$ onto the wedge $|w| \geq 0, 0 \leq \arg w \leq m\pi$ and transforms the point $z = 1$ to the point $w = 1$.

56. Evaluate : $\int_0^\infty \frac{\cos mx}{x^2+a^2}\, dx, m > 0.$

57. Define Laurent's series. Find the Laurent's series expansion of $\sin\frac{1}{z}$.

58. Classify the singularity $z = 0$ of the function $f(z) = \frac{\cosh(z^3-1)}{z^7}$ in terms of removable, pole and essential singularity. In case $z = 0$ is a pole, specify the order of the pole.

59. Evaluate the residue of the function $f(z) = \frac{\cosh(z^3-1)}{z^7}$ at $z = 0$.

60. Using part (b) $f(z) = \frac{\cosh(z^3-1)}{z^7}$, where $C: |z| = 1$ taken in the positive direction.

61. Find the value of $\oint_c \frac{cos^2(tz)}{z^3} dz\ where\ c\ is\ the\ circle |z| = 1\ and\ t > 0.$

62. Define branch and branch cut for a multi-valued function f(z).

63. Define Jordan arc with an example which is not a Jordan arc.

64. Evaluate $\int_c \frac{dz}{z}$ $,where\ c\ denotes\ the\ circle$ $|z|$=r, for any real r.

65. Find the bilinear transformation which transforms the points Z = 2,1,0 into $\omega = 1,0, i\ respectively$.

66. Evaluate $\int_0^{i+1}(x^2 - iy)dz\ \ along\ y = x.$

67. State Liouville's theorem connecting on complex analysis.

68. Find the points at which w= sin(z) is not conformal

69. When α is a fixed real number , show that the function

$$f(z) = \sqrt[3]{r}\ \ e^{\frac{i\theta}{3}} \qquad \text{(r>0, } \alpha < \theta < \alpha + 2\pi)$$

has derivative everywhere in its domain of definition.

70. Find the value of $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ by contour integration

71. Let α be a zero of $f(z)$ of order m and α be a zero of $\varphi(z)$ of order n. find the order of zero at z=α of the function $f(z)\varphi(z)$.

72. Use an antiderivative and evaluate the integral $\int_{-1-i\sqrt{3}}^{1+i\sqrt{3}}(\frac{5\pi}{z} + 3iz^{i-1})dz$ by taking any path of integration in region y<$\sqrt{3}x$ from $z = -1 - i\sqrt{3}$ to $z = 1 + i\sqrt{3}$, except for it's end points .Use principal branches of the required function.

73. Let C be the circle $|z| = \frac{3}{2}$ in the complex plane and $\int_c \left(\frac{z+1}{z^2-3z+2} + \frac{a}{z-1}\right) dz = 0$ .Then find the value of a.

74. Determine the number of roots, counting multiplicities, of the polynomial $z^4 + 3z^3 + 6$ inside $|z| = 2$.

75. Find the Taylor or Laurent series expansion of the function $f(z) = \frac{3}{z(z-i)}$ with center at z=-1 and region of convergence : 1< $|z + i| < 2$

State and prove cauchy's integral formula in connection with complex analysis.

76. Let C be a closed rectifiable Jordan curve , if $f(z)$ is analytic on I(C) except for a finite number of poles in I(C) and $f(z) \neq 0$ any where in C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$ where N and P are the number of zero's and pole's respectively of $f(z)$ inside C, each counted according to their order.
77. Use the Schwarz-chritoffel transformation to arrive at the transformation $w = z^m (0 < m < 1)$,which maps the half plane $y \geq 0$ onto the wedge $|w| \geq 0$,$0 \leq arg w \leq m\pi$ and transforms the point $z = 1$ to onto the point $w = 1$.
78. Find the fixed points of the transformation $w = \frac{z-1}{z+1}$.
79. A linear transformation with two distinct fixed points α *and* β can be put in a form $\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}$, where k is constant .Under what value's of k, the above transformation is elliptic ,hyperbolic and loxodromic?
80. Evaluate $\int_C \frac{e^z}{z^2(z+1)^3}dz$, where C : $9x^2 + 4y^2 = 36$

End